NESTED BIPARTITE GRAPHS

BY

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ABSTRACT

We investigate a class of bipartite graphs, whose structure is determined by a binary number.

Introduction

We call a graph *nested*, when among every 4 points, spanning two disjoint edges, there is at least one additional edge. We investigate the class of nested bipartite graphs and for example calculate the number of nonisomorphic ones with m points. Then we study the complete bipartite graph, in which the edges are coloured with two colours, such that every colour defines a nested partial graph. The investigation is motivated by the relation of such graphs to the combinatorial structure of a certain class of convex polytopes [2]. It raises a combinatorial problem which we cannot solve: In a binary number with m digits count the number of 'alternating' subsequences with n digits, $n \leqq m$. What is the maximal number of such subsequences?

1. The structure of nested bipartite graphs

Throughout this paper, a graph $\gamma = (M, \mathfrak{P})$ is always an *undirected graph without loops and double edges*, having the (not necessarily finite) set M as its pointset and $\mathfrak{P}$ as its set of edges. Therefore we assume

$$\mathfrak{P} \subset \{\{a,b\}:\ a,b \in M,\ a \neq b\}.$$

A bipartition $\{P,Q\}$ of M (i.e. $P \cup Q = M$, $P \cap Q = \varnothing$) is a *bipartition of* γ if

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$$\{\{a,b\}\in\mathfrak{P}:\ a,b\in P\} = \{\{a,b\}\in\mathfrak{P}: a,b\in Q\} = \varnothing .$$

Then γ is a *bipartite graph.* Furthermore

DEFINITION 1.1. $\gamma = (M,\mathfrak{P})$ is *nested,* when for every $\{a,b\}$, $\{c,d\}\in\mathfrak{P}$ at least one of the sets $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$ is in $\mathfrak{P}$, too.

LEMMA 1.1. *In a nested graph the nonisolated points form a component. Every two points of this component are joined by a chain of at most three edges.*

PROOF. When a,b are nonisolated in $\gamma = (M,\mathfrak{P})$, we have $c,d\in M$ with $\{a,c\},\{b,d\}\in\mathfrak{P}$. When $\{a,c\}\cap\{b,d\}\neq\varnothing$, then a, b are joint by a chain of length 2 or less. Otherwise the nested property gives us an additional edge among these points and so a chain of length 3 or less between a and b. $\square$

To $a\in M$ in $\gamma = (M,\mathfrak{P})$ we denote

$$\mathfrak{P}(a):= \{b\in M:\{a,b\}\in\mathfrak{P}\}. \tag{1.1}$$

LEMMA 1.2. *Suppose $\gamma = (M,\mathfrak{P})$ is bipartite and $\{P,Q\}$ is a bipartition. Then γ is nested if and only if*

$$\mathfrak{P}(a)\subset\mathfrak{P}(b) \textit{ or } \mathfrak{P}(b)\subset\mathfrak{P}(a) \textit{ for every } a,b\in P. \tag{1.2}$$

PROOF. When (1.2) is violated, we get $c\in\mathfrak{P}(a)\setminus\mathfrak{P}(b)$, $d\in\mathfrak{P}(b)\setminus\mathfrak{P}(a)$, i.e. $\{a,c\}\in\mathfrak{P}$, $\{b,c\}\notin\mathfrak{P}$, $\{b,d\}\in\mathfrak{P}$, $\{a,d\}\notin\mathfrak{P}$. Furthermore $\{a,b\}$, $\{c,d\}\notin\mathfrak{P}$, for $\{P,Q\}$ is a bipartition of γ. So γ is not nested.

Conversely, when γ is not nested, we find $\{a,c\}$, $\{b,d\}\in\mathfrak{P}$ with $\{a,b\}$, $\{a,d\}$, $\{b,c\}$, $\{c,d\}\notin\mathfrak{P}$. We may assume $a,b\in P$, so $c\in\mathfrak{P}(a)\setminus\mathfrak{P}(b)$, $d\in\mathfrak{P}(b)\setminus\mathfrak{P}(a)$, .e. (1.2) is violated. $\square$

PROPOSITION 1.1. *Suppose M is a linear ordered set having the order relation $<$ and $\{P,Q\}$ is a bipartition of M, furthermore*

$$\mathfrak{P}:= \{\{a,b\}:\ a\in P,\ b\in Q,\ a<b\}. \tag{1.3}$$

Then $\gamma:=(M,\mathfrak{P})$ is a nested bipartitite graph with the bipartition $\{P,Q\}$.

PROOF. $\{P,Q\}$ is a bipartition of γ according to (1.3). For the nested property take a, $c\in P$, b, $d\in Q$ with $\{a,b\}$, $\{c,d\}\in\mathfrak{P}$. So $a<b$, $c<d$. When $a\leqq c$, then $a<d$, so $\{a,d\}\in\mathfrak{P}$; when $c<a$, then $c<b$, so $\{b,c\}\in\mathfrak{P}$. $\square$

We want to show, that *every* nested bipartite graph $\gamma = (M, \mathfrak{P})$ may be obtained in this way. So we have to construct a suitable linear order relation on M.

DEFINITION 1.2. Suppose $\gamma = (M, \mathfrak{P})$ is bipartite, $\{P, Q\}$ is a bipartition and a, $b \in M$. Then a, b are *equivalent* iff a, $b \in P$ or $a, b \in Q$ and $\mathfrak{P}(a) = \mathfrak{P}(b)$. We write $a \sim b$, $\bar{a}$ is the equivalence class of a. The equivalence classes are called *blocks*, more specific $\bar{a}$ is a *P-block* if $a \in P$ and a *Q-block* otherwise. Furthermore

$$\bar{P}: = \{\bar{a}: a \in P\}, \ \bar{Q}: = \{\bar{b}: b \in Q\}, \ \bar{M}: = \bar{P} \cup \bar{Q}$$
$$\overline{\mathfrak{P}}: = \{\{\bar{a}, \bar{b}\}: \{a, b\} \in \mathfrak{P}\}, \ \bar{\gamma}: = (\bar{M}, \overline{\mathfrak{P}}).$$

Obviously $\bar{\gamma}$ is again a bipartite graph having the bipartition $\{\bar{P}, \bar{Q}\}$ and $\bar{\bar{\gamma}}$ is isomorphic to $\bar{\gamma}$. The equivalence relation on M depends not only on γ, but also on the bipartition $\{P, Q\}$. However, this ambiguousity is only relevant for the isolated points, for when $\mathfrak{P}(a) \neq \emptyset$, then $\mathfrak{P}(a) = \mathfrak{P}(b)$ implies a, $b \in P$ or a, $b \in Q$.

Now we are prepared to define a linear order on M having the properties of Proposition 1.1:

DEFINITION 1.3. Suppose $\gamma = (M, \mathfrak{P})$ is nested, bipartite and $P \subset M$ such that $\{P, Q\}$ is a bipartition of γ. We define a P-relation $<$ on M as follows:

i) On every block $\bar{a} \in \bar{M}$ the relation $<$ is a linear order.

Now suppose a, $b \in M$ are not equivalent. Then $a < b$ holds if and only if one of the following conditions is satisfied:

ii) $a, b \in Q, \ \mathfrak{P}(a) \subset \mathfrak{P}(b)$

iii) $a, b \in P, \ \mathfrak{P}(b) \subset \mathfrak{P}(a)$

iv) $a \in Q, \ b \in P, \ \{a, b\} \notin \mathfrak{P}$

v) $a \in P, \ b \in Q, \ \{a, b\} \in \mathfrak{P}$.

LEMMA 1.3. *A P-relation $<$ develops canonically from M upon $\bar{M}$ and defines a $\bar{P}$-relation $\lessdot$.*

PROOF. For $\bar{a}, \bar{b} \in \bar{P}$, $\bar{a} = \bar{b}$, define $\bar{a} \lessdot \bar{b}$ iff there exists $a \in \bar{a}$, $b \in \bar{b}$ with $a < b$. Then $a' < b'$ for *every* $a' \in \bar{a}$, $b' \in \bar{b}$. Therefore $\lessdot$ has the properties $(i), \cdots (v)$ of Definition 1.3. □

A P-relation is not uniquely defined to a given P, for it is an arbitrary linear order within every equivalence class $\bar{a}$. However, this is obviously the only ambiguousity:

LEMMA 1.4. *When* $<_i$ $(i = 1,2)$ *are P-relations, then* $\lesssim_1 = \lesssim_2$.

PROOF. Obvious from Definition 1.3. □

LEMMA 1.5. *When* $<$ *is a P-relation, then the inverse relation* $>$ *is a Q-relation.*

PROOF. This is again an immediate consequence of Definition 1.3. □

THEOREM 1.1. *If* $\gamma = (M, \mathfrak{P})$ *is a nested bipartite graph, then every P-relation is a linear order on M and*

$$\mathfrak{P} = \{\{a,b\}:\ a \in P,\ b \in Q,\ a < b\}. \tag{1.4}$$

PROOF. (1.4) is obvious from (iv), (v) of Definition 1.3. So we only have to show that $<$ is a linear order on M. We check first, that $<$ is antisymmetric. Suppose $a, b \in P$ or $a, b \in Q$. On $\bar{a}$ our relation is a linear order by Definition 1.3, (i). We may therefore assume, that a, b are not equivalent. From (1.2) we obtain

$$\mathfrak{P}(a) \underset{\neq}{\subset} \mathfrak{P}(b) \quad \text{or} \quad \mathfrak{P}(b) \underset{\neq}{\subset} \mathfrak{P}(a).$$

Therefore, when a, $b \in P$, the first relation gives us $b < a$, not $a < b$, the second leads to $a < b$, not $b < a$. When $a, b \in Q$, we get $a < b$ not $b < a$ from the first and $b < a$, not $a < b$ from the second relation. When $a \in Q$, $b \in P$, then $a < b$, not $b < a$ iff $\{a, b\} \notin \mathfrak{P}$ and $b < a$, not $a < b$ otherwise.

Now we show, that $<$ is transitive. Suppose $a < b$, $b < c$ we have to check $a < c$ in all the possible cases.

1) $a \sim b$. When $b \sim c$, too, then $a < c$ follows from definition 3, (i). Now assume, that b, c are not equivalent. When b, $c \in Q$, then $b < c$ gives us $\mathfrak{P}(b) \subset_{\neq} \mathfrak{P}(c)$, so from $a, b \in Q$ and $\mathfrak{P}(a) = \mathfrak{P}(b)$ we have $\mathfrak{P}(a) \subset_{\neq} \mathfrak{P}(b)$, $a < c$. When $b, c \in P$, then $\mathfrak{P}(c) \subset_{\neq} \mathfrak{P}(b)$ so again $a < c$.

2) $a, b \in Q$ *not equivalent.* So we have $\mathfrak{P}(a) \subset_{\neq} \mathfrak{P}(b)$. Therefore when $c \in Q$, then $b < c$ gives us $\mathfrak{P}(b) \subset \mathfrak{P}(c)$, so $\mathfrak{P}(a) \subset_{\neq} \mathfrak{P}(b)$, therefore $a < c$. When $c \in P$ then $b < c$ gives us $\{b, c\} \notin \mathfrak{P}$ according to Definition 1.3, (iv). From $\mathfrak{P}(a) \subset \mathfrak{P}(b)$ we have $\{a, c\} \notin \mathfrak{P}$, so $a < c$.

3) $a, b \in P$, *not equivalent.* So $\mathfrak{P}(b) \subset_{\neq} \mathfrak{P}(a)$. Again $c \in P$ gives us $\mathfrak{P}(c) \subset \mathfrak{P}(b)$, so $a < c$. When $c \in Q$, then $b < c$ is equivalent to $\{b, c\} \in \mathfrak{P}$, so $\mathfrak{P}(b) \subset \mathfrak{P}(a)$ gives us $\{a, c\} \in \mathfrak{P}$ and therefore again $a < c$.

4) $a \in Q$, $b \in P$. Here $\{a, b\} \notin \mathfrak{P}$. When $c \in P$, then $\mathfrak{P}(c) \subset \mathfrak{P}(b)$ and $a < c$

follows from $a \in Q$, $c \in P$, $\{a,c\} \notin \mathfrak{P}$. When $c \in Q$, then $\{b,c\} \in \mathfrak{P}$, so $b \notin \mathfrak{P}(a)$, $b \in \mathfrak{P}(c)$ implies $\mathfrak{P}(a) \subset_{\neq} \mathfrak{P}(b)$ according to (1.2), therefore $a < c$.

5) $a \in P$, $b \in Q$. So $\{a,b\} \in \mathfrak{P}$. Again for $c \in Q$ the relation $b < c$ gives us $\mathfrak{P}(b) \subset \mathfrak{P}(c)$, so $\{a,c\} \in \mathfrak{P}$ and therefore $a < c$. When $c \in P$, then $\{b,c\} \notin \mathfrak{P}$, so $b \in \mathfrak{P}(a)$, $b \notin \mathfrak{P}(c)$, i.e. $\mathfrak{P}(c) \subset \mathfrak{P}(a)$ and $a < c$. □

We use this theorem for obtaining a representation of every finite nested bipartite graph by a binary number:

Definition 1.4. Suppose $d = (d_1, \cdots, d_m)$ is a binary number, define $M := \{1,2,\cdots m\}$

$$\text{(1.5)} \qquad \begin{cases} P := \{i \in M\colon d_i = 0\} \\ Q := \{j \in M\colon d_j = 1\} \end{cases}$$

$$\text{(1.6)} \qquad \mathfrak{P} := \{\{i,j\}\colon d_i = 0,\ d_j = 1,\ i < j\}.$$

Then $\gamma(d) := (M, \mathfrak{P})$ is the *graph of* d.

Corollary 1.1. *The graph $\gamma(d)$ is nested, bipartite and to every finite nested bipartite graph γ there exists a binary number d such that γ and $\gamma(d)$ are isomorphic.*

Proof. This is an immediate consequence of Proposition 1.1 and Theorem 1.1. □

A point $a \in M$ is a *universal point* of γ, if $a \in P$ and $\mathfrak{P}(a) = Q$ or $a \in Q$ and $\mathfrak{P}(a) = P$.

Corollary 1.2. *Suppose $\gamma = (M, \mathfrak{P})$ is nested and bipartite having the bipartition $\{P, Q\}$ and $<$ is a P-relation.*

i) *γ has an isolated point in Q if and only if $\bar{M}$ has a minimal Q-block. (Minimal with respect to the $\bar{P}$-relation $\lessdot$ induced on $\bar{M}$ by the P-relation $<$ according to Lemma* 1.3).

ii) *γ has an isolated point in P if and only if $\bar{M}$ has a maximal P-block.*

iii) *γ has a universal point in Q if and only if $\bar{M}$ has a maximal Q-block.*

iv) *γ has a universal point in P if and only if $\bar{M}$ has a minimal P-block.*

v) *γ is connected if and only if $\bar{M}$ has no minimal Q-block and no maximal P-block.*

Proof. i): $b \in Q$ is isolated iff there exists no $a \in P$ with $a < b$, i.e. iff $\bar{b}$ is minimal. The cases ii), iii), iv) follows by an analogous argument.

(v) From (i), (ii) we obtain that the condition: $\bar{M}$ has no minimal Q-block and no maximal P-block is equivalent to: M has no isolated points. Therefore (v) follows from Lemma 1.1. □

When M is finite, the condition (i) of Corollary 1.2 may be replaced by: γ has an isolated point in Q if and only if M has a minimal element in Q with respect to the given P-relation $<$. The conditions (ii)–(v) may be changed analogously.

COROLLARY 1.3. *For a binary number $d = (d_1, \cdots, d_m)$ the graph $\gamma(d)$ is connected if and only if $d_1 = 0$, $d_m = 1$.*

PROOF. Definition 1.4 and Corollary 1.2, (v). □

COROLLARY 1.4. *The nested, bipartite graphs $\gamma_i = (M_i, \mathfrak{P}_i)$ are isomorphic if and only if there exist P_i-relations $<_i$ on M_i and an order-isomorphism ϕ from M_1 to M_2 with $\phi(P_1) = P_2$.*

PROOF. Suppose first, that γ_1, γ_2 are isomorphic graphs and that ψ is a graph-isomorphism from γ_1 onto γ_2. From Theorem 1.1 we get a P_1-relation $<_1$ on M_1 satisfying (1.4). Then $\{\psi(P_1), \psi(M_1 \setminus P_1)\}$ is a bipartition of γ_2 and the definition: $a_2 <_2 b_2$ if and only if $\psi^{-1}(a_2) <_1 \psi^{-1}(b_2)$ for every a_2, $b_2 \in M_2$ gives us a $\psi(P_1)$-relation $<_2$ on M_2 which is isomorphic to $<_1$.

Conversely, when ϕ is an order isomorphism with $\phi(P_1) = P_2$, then (1.4) gives us $\phi(\mathfrak{P}_1) = \mathfrak{P}_2$. □

DEFINITION 1.5. $\mathbb{D}_m$ denotes the set of all binary numbers with m digits. We write $0' := 1$, $1' := 0$ and to $d = (d_1, \cdots, d_m) \in \mathbb{D}_m$ define $d' := (d_1', \cdots, d_m')$, $d^- := (d_m, \cdots, d_1)$.

So d', $d^- \in \mathbb{D}_m$, too and $(d')^- = (d^-)'$.

COROLLARY 1.5. *Suppose $d, d^* \in \mathbb{D}_m$ and $\gamma(d)$ is connected, (so when $d = (d_1, \cdots, d_m)$, then $d_1 = 0$, $d_m = 1$ by Corollary 1.3). Then $\gamma(d)$, $\gamma(d^*)$ are isomorphic if and only if $d^* = d$ or $d^* = (d^-)'$.*

PROOF. Assume first $d^* = d'^-$. Then (1.5) is a bipartition for both $\gamma(d)$ and $\gamma(d^*)$ and the mapping ϕ: $\phi(i) = m - i$ $(i = 1, \cdots m)$ has the properties of Corollary 1.4. Conversely, when $\gamma(d)$ is connected, then the bipartition $\{P, Q\}$ is uniquely defined either leading to d or d'^-. □

2. The number of nested bipartite graphs with m points

THEOREM 2.1. *Suppose p, q are positive integers, $p + q = m$ and $\beta(p, q)$ is the cardinality of nonisomorphic nested, bipartite connected graphs having he bipartition $\{P, Q\}$ with $\# P = p$, $\# Q = q$.*

Then

$$(2.1)\qquad \beta(p,q) = \begin{cases} \binom{m-2}{p-1} & \text{when } p \neq q, \\ \frac{1}{2}\binom{m-2}{\frac{m-2}{2}} + 2^{\frac{1}{2}(m-4)}, & \text{when } p = q = m/2. \end{cases}$$

PROOF. Corollary 1.5 gives us

$$\beta(p,q) = \#\{\{d, d'^{-}\}: d = (d_1, \cdots, d_m) \in \mathbb{D}_m,\ d_1 = 0,\ d_m = 1,\ \sum_{i=1}^{m} d_i = q\}.$$

When $p \neq q$, then

$$\beta(p,q) = \#\{d \in \mathbb{D}_m: d_1 = 0,\ d_m = 1,\ \Sigma\, d_i = q\} = \binom{m-2}{p-1}.$$

When $p = q = \frac{m}{2}$, then $\binom{m-2}{\frac{m-2}{2}}$ counts the sets $\{d, d'^{-}\}$ with $d \neq d'^{-}$ twice. Now for

$$d = (d_1, \cdots d_{m/2},\ d_{m/2+1}, \cdots d_m)$$

we have $d = d'^{-}$ if and only if $d_{m/2-j+1} = d'_{m/2+j}$ for $j = 1, \cdots m/2$. Therefore

$$\#\ \{\{d, d'^{-}\}: d \in \mathbb{D}_m,\ d_1 = 0,\ d_m = 1,\ d = d'^{-}\}$$

$$= \#\ \{(d_1, \cdots d_{m/2}) \in \mathbb{D}_{m/2}: d_1 = 0\} = 2^{(m-2)/2}.$$

This gives us

$$\beta\left(\frac{m}{2}, \frac{m}{2}\right) = \frac{1}{2}\left[\binom{m-2}{\frac{m-2}{2}} - 2^{(m-2)/2}\right] + 2^{(m-2)/2} = \frac{1}{2}\binom{m-2}{\frac{m-2}{2}} + 2^{\frac{1}{2}(m-4)}. \quad \square$$

COROLLARY 2.1. *The cardinality $\beta(m)$ of nonisomorphic nested bipartite connected graphs with m points is*

$$(2.2)\qquad \beta(m) = \begin{cases} 2^{m-3} & \text{when } m \equiv 1(2), \\ 2^{m-3} + 2^{m/2-2}, & \text{when } m \equiv 0(2). \end{cases}$$

PROOF. $\beta(m) = \Sigma\, \{\beta(p,q): p + q = m,\ 1 \leqq p \leqq [\frac{1}{2}m]\}$. Therefore, when $m \equiv 1(2)$, Theorem 2.1 gives us

$$\beta(m) = \sum_{p=1}^{\frac{1}{2}(m-1)} \binom{m-1}{p-1} = \frac{1}{2}\sum_{v=0}^{m-2} \binom{m-2}{v} = 2^{m-3}.$$

When $m \equiv 0(2)$ we obtain

$$\beta(m) = \sum_{p=1}^{\frac{1}{2}(m-1)} \binom{m-1}{p-1} + \frac{1}{2}\binom{m-2}{\frac{m-2}{2}} + 2^{m/2-2}$$

$$= \frac{1}{2}\sum_{v=0}^{m-2}\binom{m-2}{v} - \frac{1}{2}\binom{m-2}{\frac{m-2}{2}} - \frac{1}{2}\binom{m-2}{\frac{m-2}{2}} + 2^{m/2-2}$$

$$= 2^{m-3} + 2^{(m-4)/2}. \quad \square$$

3. Relations between nested bipartite graphs

DEFINITION 3.1. An edge $\{a,b\}$ of a nested bipartite graph $\gamma = (M,\mathfrak{P})$ is an *endedge*, if $(M,\mathfrak{P}\setminus\{\{a,b\}\})$ is nested and bipartite, too.

PROPOSITION 3.1. *Suppose $\{a,b\}$ is an edge of the nested bipartite graph $\gamma = (M,\mathfrak{P})$, $\{P,Q\}$ is a bipartition of γ, $a \in P$, $b \in Q$ and $<$ is a P-relation. Then $\{a,b\}$ is an endedge if and only if the following condition holds:*

(*) $\{c,d\} \in \mathfrak{P}$, $c \in P$, $d \in Q$ *and* $\bar{a} < \bar{c}$ *implies* $\bar{b} < \bar{d}$ *or* $\bar{b} = \bar{d}$.

PROOF. a) Suppose $\{a,b\}$ is not an endedge. Then we find $\{a_1,b_2\}$, $\{a_2,b_1\} \in \mathfrak{P} \setminus \{a,b\}$ with $a_1,a_2 \in P, b_1,b_2 \in Q$ and $\{a_1,b_1\}$, $\{a_2,b_2\} \notin \mathfrak{P} \setminus \{a,b\}$, for $(M,\mathfrak{P} \setminus \{\{a,b\}\})$ is not nested. γ is nested, therefore $\{a_1,b_1\} = \{a,b\}$ or $\{a_2,b_2\} = \{a,b\}$. We assume $\{a_2,b_2\} = \{a,b\}$, so $a_2 = a$, $b_2 = b$. We have $b_1 < a_1$ for $\{a_1,b_1\} \notin \mathfrak{P}$, $a_1 < b$, $a < b_1$ for $\{a_1,b\}$, $\{a,b_1\} \in \mathfrak{P}$ together $a < b_1 < a_1$ and $b_1 < a_1 < b$, i.e. $\{a_1,b_1\}$ violates the condition (*).

b) Conversely, let us assume that $\{a,b\} \in \mathfrak{P}$ with $a \in P$, $b \in Q$ violates (*). Then we find $\{c,d\} \in \mathfrak{P}$, $c \in P$, $d \in Q$ with $\bar{a} < \bar{b}$ and $\bar{d} < \bar{c}$. From $\bar{a} < \bar{c}$ we obtain $u \in \mathfrak{P}(a) \setminus \mathfrak{P}(c)$ according to Definition 3, (iii). From $\bar{d} < \bar{b}$ and $(c,d\} \in \mathfrak{P}$ we get $\{c,b\} \in \mathfrak{P}$. So in $(M,\mathfrak{P} \setminus \{\{a,b\}\}$ we have $\{a,u\}$, $\{c,b\} \in \mathfrak{P} \setminus \{\{a,b\}\}$, with $\{c,u\}$, $\{a,b\} \notin \mathfrak{P} \setminus \{\{a,b\}\}$, proving that this graph is not nested. $\square$

COROLLARY 3.1. *When $\gamma = (M,\mathfrak{P})$ is nested, bipartite, $\mathfrak{P} \neq \varnothing$ and $\bar{\gamma}$ is finite, then γ has an endedge.*

PROOF. When $\{a,b\}$ is not endedge, we find $\{c,d\} \in \mathfrak{P}$ with $\bar{c} < \bar{a}$, $\bar{d} < \bar{b}$. Continuing this argument leads to an endedge, for $\bar{\gamma}$ is finite. $\square$

When M is the set of rational numbers in the open unit interval and P is a subset of M such that P and $M \setminus P$ are both dense in M, define

$\mathfrak{P}: = \{\{a,b\}: a \in P,\ b \in M \setminus P,\ a < b\}$. Then $\gamma = (M,\mathfrak{P})$ is a nested bipartite graph without endedges.

COROLLARY 3.2. *When $\gamma = (M,\mathfrak{P})$ is a finite nested bipartite graph then there exists a sequence*

$$\emptyset = \mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_s = \mathfrak{P} \tag{3.1}$$

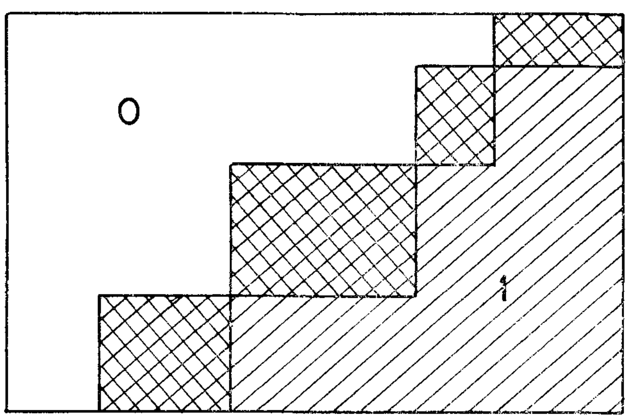
with $\#(\mathfrak{P}_i \setminus \mathfrak{P}_{i-1}) = 1$ for $i = 1, \cdots s$ such that every $\gamma_i = (M,\mathfrak{P}_i)$ is nested and bipartite.

PROOF. Obvious from Corollary 3.1. □

COROLLARY 3.3. *When $d \in \mathbb{D}_m$, then $\{u,v\} \in \mathfrak{P}$, $v \in Q$, is an endedge of $\gamma(d)$ if and only if there exists a $w \in M = \{1,\cdots m\}$ with $d_u = d_{u+1} = \cdots = d_w = 0$, $d_{w+1} = \cdots = d_v = 1$. In this case define $d^* = (d_1,\cdots d_{w-1}, 1, 0, d_{w+2},\cdots d_m)$. Then $\gamma(d^*)$ is isomorphic to $\gamma^* = (M,\mathfrak{P} \setminus \{\{u,v\}\})$.*

PROOF. Apply Definition 1.4 and Corollary 1.1. □

When $\gamma = (M,\mathfrak{P})$ is a finite nested bipartite graph, then its endedges may be easily obtained from its adjacency matrix. They are cross-hatched in the following diagram:



(From Lemma 1.2 we obtain that to every nested bipartite graph there exists an adjacency matrix of the form of the diagram above).

DEFINITION 3.7. When $\gamma = (M,\mathfrak{P})$ has the bipartition $\{P,Q\}$ and

$$Q: = \{\{a,b\}: a \in P,\ b \in Q,\ \{a,b\} \in \mathfrak{P}\} \tag{3.2}$$

then $\gamma^c = (M,\mathfrak{P})$ is the $\{P,Q\}$-*complement* of γ.

LEMMA 3.1. *When γ is nested, then γ^c is nested, too.*

PROOF. Suppose $\{a,b\}$, $\{c,d\}\in\mathfrak{Q}$, a, $c\in P$, b, $d\in Q$. Then $\{a,d\}\in\mathfrak{P}$ and $\{c,b\}\in\mathfrak{P}$ is impossible, for γ is nested. So $\{a,d\}\in\mathfrak{Q}$ or $\{c,b\}\in\mathfrak{Q}$. □

The definition of γ^c depends not only on γ but also on the bipartition $\{P,Q\}$. This bipartition and therefore γ^c is uniquely determined if and only if γ is connected. According to Corollary 1.2 in this case γ^c is not connected and therefore has more than one complement.

The following two corollaries are an immediate consequence of Lemma 3.1:

COROLLARY 3.4. *When γ is nested having the bipartition $\{P,Q\}$ and the $\{P,Q\}$-complement γ^c, then every P-relation of γ is a Q-relation of γ^c. So*

$$\mathfrak{Q} = \{\{a,b\}:\ a\in P,\ b\in Q,\ b<a\} \tag{3.3}$$

COROLLARY 3.5. $\gamma(d)^c = \gamma(d')$ *for every* $d\in\mathbb{D}_m$.

Another consequence is the following extension of (3.1) in Corollary 3.2.

COROLLARY 3.6. *When $\gamma=(M,\mathfrak{P})$ is finite nested and bipartite with the bipartition $\{P,Q\}$ and*

$$\mathfrak{M} = \{\{a,b\}: a\in P,\ b\in Q\} \tag{3.4}$$

then there exists a sequence

$$\varnothing = \mathfrak{P}_0\subset\mathfrak{P}_1\subset\cdots\subset\mathfrak{P}_s = \mathfrak{P}\subset\mathfrak{P}_{s+1}\subset\cdots\subset\mathfrak{P}_t = \mathfrak{M} \tag{3.5}$$

such that every $\gamma_i=(M,\mathfrak{P}_i)$ is nested, bipartite with the same bipartition $\{P,Q\}$ and $\#(\mathfrak{P}_i\setminus\mathfrak{P}_{i-1})=1$ for $i=1,\cdots t$.

PROOF. Apply Corollary 3.2 on γ and on γ^c. □

4. Alternating chains in the nested bicoloured, complete bipartite graph

Let $\omega=(M,\mathfrak{M})$ denote the complete bipartite graph with the bipartition $\{P,Q\}$ so that $\mathfrak{M}$ is given by (3.4) and with the bicolouring $\{\mathfrak{P},\mathfrak{Q}\}$ of $\mathfrak{M}$. (A bicoloring is simply a bipartition of $\mathfrak{M}$). A colour $\mathfrak{P}$ (or $\mathfrak{Q}$) of ω is *nested*, if $(M,\mathfrak{P})$ (or $(M,\mathfrak{Q})$) is a nested graph. From Lemma 3.1 we learn that either $\mathfrak{P}$ and $\mathfrak{Q}$ are both nested or both not nested. Furthermore, when $d\in\mathbb{D}_m$, then $\omega(d)$ is the nested coloured bipartite complete graph of d having the bipartition (1.5) and the nested bicolourings

$$\begin{cases}\mathfrak{P}(d):\ =\ \{\{i,j\};\ d_i=0,\ d_j=1,\ i<j\}\\ \mathfrak{Q}(d):\ =\ \{\{i,j\}:\ d_i=1,\ d_j=0,\ i<j\}\end{cases} \tag{4.1}$$

A chain $\nu = (N, \mathfrak{N})$ *in* ω is *alternating*, iff every 2 adjacent edges of ν have different colours.

LEMMA 4.1. *When ω has an alternating cyclic chain, then ω has an alternating cyclic chain of length 4.*

PROOF. When $\nu = (N, \mathfrak{N})$ is an alternating cyclic chain of minimal length $2t$, we have an arrangement of $\mathfrak{N}$ to a sequence

$$(\{a_1, b_1\}, \{b_1, a_2\}, \{a_2, b_2\}, \cdots \{b_{t-1}, a_t\}, \{a_t, b_t\}, \{b_t, a_1\})$$

where $a_i \in P$, $b_i \in Q$ for $i = 1, \cdots, t$ and $\{a_1, b_1\} \in \mathfrak{P}$. When $t > 2$, then $\{a_2, b_t\} \notin \mathfrak{N}$. When $\{a_2, b_t\} \in \mathfrak{Q}$, then $(\{a_2, b_2\}, \{b_2, a_3\}, \cdots, \{b_{t-1}, a_t\}, \{a_t, b_t\}, \{b_t, a_2\}$ defines an alternating cyclic chain of length $2t - 2$. When $\{a_2, b_t\} \in \mathfrak{P}$, then $(\{a_1, b_1\}, \{b_1, a_2\}, \{a_2, b_t\}, \{b_t, a_1\})$ defines an alternating cyclic chain of length 4. Both results contradict the definition of t. □

PROPOSITION 4.1. *The colouring $\{\mathfrak{P}, \mathfrak{Q}\}$ of ω is nested if and only if ω has no alternating cyclic chain.*

PROOF. The colouring is not nested if and only if we find $a_1, a_2 \in P$, $b_1, b_2 \in Q$ with $\{a_1, b_1\}, \{a_2, b_2\} \in \mathfrak{P}$, $\{a_1, b_2\}, \{a_2, b_1\} \in \mathfrak{Q}$, i.e. if there exists an alternating cyclic chain of length 4. Lemma 4.1 completes the proof. □

PROPOSITION 4.2. *If $\omega = (M, \mathfrak{M})$ is bicoloured by $\{\mathfrak{P}, \mathfrak{Q}\}$, such that $\gamma = (M, \mathfrak{P})$ is nested with the bipartition $\{P, Q\}$, $<$ is a P-relation on M and $\nu = (N, \mathfrak{N})$ is a chain in ω, then ν is alternating if and only if the monotone arrangement of N to a sequence $(i_1, \cdots, i_r)$ with $i_1 < \cdots < i_r$ has the property*

$$\mathfrak{N} = \{\{i_\nu, i_{\nu+1}\} : \nu = 1, \cdots, r-1\}. \tag{4.2}$$

PROOF. We arrange N to a sequence $(j_1, \cdots, j_r)$ with $\mathfrak{N} = \{\{j_\nu, j_{\nu+1}\}: \nu = 1, \cdots, r-1\}$ and $j_1 < j_r$. This arrangement is uniquely determined and Definition 1.3 gives us immediately $j_1 < \cdots < j_r$ if and only if ν is alternating. □

In general, a chain is uniquely determined by its edgeset, but not by its pointset. However from Proposition 4.2 we obtain directly

COROLLARY 4.1. *When ω is nested coloured and $\nu_i = (N_i, \mathfrak{N}_i)$, $i = 1,2$ are alternating chains with $N_1 = N_2$, then $\nu_1 = \nu_2$.*

PROOF. From (4.2) we have $\mathfrak{N}_1 = \mathfrak{N}_2$. □

DEFINITION 4.1. A subsequence

$$\delta = (d_{i_1}, \cdots, d_{i_r}) \tag{4.3}$$

of $d = (d_1, \cdots, d_m) \in \mathbb{D}_m$ is *alternating* if and only if

$d_{i_\nu} + d_{i_{\nu+1}} = 1$ for $\nu = 1, \cdots r-1$.

COROLLARY 4.2. *δ is alternating if and only if $\{i_1, \cdots, i_r\}$ is the pointset of an alternating chain of $\omega(d)$.*

PROOF. The condition $d_{i_\nu} + d_{i_{\nu+1}} = 1$ is equivalent to: $i_\nu \in P$ iff $i_{\nu+1} \in Q$. □

5. Counting alternating chains

We now prepare a formula which counts the number of alternating chains in $\omega(d)$: When m is a positive integer, then

$$\nu = (n_1, \cdots, n_t): n_i \text{ positive integers}, \ n_1 + \cdots + n_t = m \tag{5.1}$$

is a *partition* of m.

DEFINITION 5.1. The *partition $\nu(d)$ of d*, $d \in \mathbb{D}_m$, is defined by

$$\nu(d) = (n_1(d), \cdots n_t(d)), \tag{5.2}$$

where the $n_p(d)$ are defined inductively as follows:

$$n_1(d) := \# \{i \varepsilon \{1, 2, \cdots m\}: d_i = d_{i-1} = \cdots = d_1\}.$$

When $p > 1$, $n_1 + \cdots + n_{p-1} < m$ then

$$n_p(d) := \# \{i \varepsilon \{1, 2, \cdots, m\}: d_i = d_{i-1} = \cdots = d_{n_{p-1}+1}\}.$$

The following proposition is plain:

PROPOSITION 5.1. *For every $d \in \mathbb{D}_m$ the partition $\nu(d)$ of d is a partition of m. When d, $d^* \in \mathbb{D}_m$, then $\nu(d) = \nu(d^*)$ if and only if $d = d^*$ or $d = d'$. Conversely every partition ν of m defines exactly two binary numbers d, $d^* \in \mathbb{D}_m$ with $\nu(d) = \nu(d^*)$ and here $d^* = d'$.*

When $\{\mathfrak{P}, \mathfrak{Q}\}$ is a nested bicolouring of the bipartite complete graph $\omega = (M, \mathfrak{M})$ then the nested subgraphs $\gamma = (M, \mathfrak{N})$ and $\gamma^c = (M, \mathfrak{Q})$ of ω define the same partition of M in blocks. We call these blocks *the blocks of ω with respect to the nested bicolouring* $\{\mathfrak{P}, \mathfrak{Q}\}$. Now, when ω is finite and $m = \# M$, we have a binary number $d \in \mathbb{D}_m$ such that ω and $\omega(d)$ are isomorphic coloured. From Definition 5.1 we obtain

LEMMA 5.1. *Suppose $(N_1, \cdots, N_t)$ are the blocks of $\omega(d)$ and $N_1 < \cdots < N_t$. Then $\# N_i = n_i(d)$.*

DEFINITION 5.2. Suppose $d \in \mathbb{D}_m$ and

$$\eta = (h_1, \cdots, h_r),\ 1 \leqq h_1 < \cdots < h_r \leqq m. \tag{5.3}$$

When $\xi = (g_1, \cdots, g_s)$, $1 \leqq g_1 < \cdots < g_s \leqq m$, then η, ξ are *d-equivalent*, iff $s = r$ and h, g_i are in the same block of $\omega(d)$ for $i = 1, \cdots, r$.

So, when $h_i \leqq g_i$, then the d-equivalence requires $d_{h_i} = d_{h_i+1} = \cdots = d_{g_i}$.

LEMMA 5.2. *To $d \in \mathbb{D}_m$ let*

$$(N_1, \cdots, N_t),\ N_1 < \cdots < N_t \tag{5.4}$$

be the increasing sequence of blocks of $\omega(d)$. Furthermore for (5.3) *assume $h_i \in N_{f(h_i)}$ for $i = 1, \cdots, r$. Then the cardinality of sequences ξ, which are d-equivalent to η is $n_{f(h_i)} \cdot \cdots \cdot n_{f(h_r)}$.*

PROOF. To h_i we may choose g_i arbitrarily in $N_{f(h_i)}$. □

LEMMA 5.3. *The set $\{h_1, \cdots, h_r\}$ of elements of η in* (5.3) *is the pointset of an alternating chain of $\omega(d)$ if and only if $f(h_i) + f(h_{i+1}) \equiv 1(2)$ for $i = 1, \cdots, r-1$.*

PROOF. The condition is equivalent to $d_{h_i} + d_{h_{i+1}} = 1$ for $i = 1, \cdots, s-1$. The lemma follows now from Definition 4.1. □

This leads us to the following notation

$$\mathfrak{A}_{r\,t} := \{I = \{i_1, \cdots, i_r\} \subset \{1, 2, \cdots, t\} : i_1 < \cdots < i_r, i_\nu + i_{\nu+1} \equiv 1(2) \text{ for } \nu = 1, \cdots, r-1\}. \tag{5.5}$$

PROPOSITION 5.2. *When $d \in \mathbb{D}_m$ has the sequence of blocks* (5.4) *and $n_i = \# N_i$ and when $a_r(d)$ is the cardinality of all alternating subsequences of d with r digits, then*

$$a_r(d) = \sum_{I \in \mathfrak{A}_{r,t}} \prod_{i \in I} n \tag{5.6}$$

PROOF. Apply Lemma 5.2 and Lemma 5.3. □

(5.6) suggest to investigate the following polynomial:

DEFINITION 5.3.

$$A_r(x_1, \cdots, x_t) = \sum_{I \in \mathfrak{A}_{r,t}} \prod_{i \in I} x_i$$

is the *alternating polynomial.*

Define

$$\begin{cases} b_{r,t} := \# \{I \in \mathfrak{A}_{r,t} : \min I \equiv 1(2)\} \\ c_{r\,t} := \# \{I \in \mathfrak{A}_{r,t} : \min I \equiv 0(2)\}. \end{cases} \tag{5.8}$$

LEMMA 5.4. *When* $t > 0$, *then*

$$c_{r\,t} = b_{r,t-1}. \tag{5.9}$$

PROOF. The mapping $i \to i-1$ for $i = 2,\cdots,t$ maps $\{I \in \mathfrak{A}_{r,t}: \min I \equiv 0(2)\}$ one-to-one onto $\{I^* \in \mathfrak{A}_{r,t-1}: \min I^* \equiv 1(2)\}$. □

LEMMA 5.5. *When* $t \geqq r+2$ *and* $r > 1$, *then*

$$b_{r,t} = b_{r,t-2} + b_{r-1,t-1}. \tag{5.10}$$

PROOF. $\{I \in \mathfrak{A}_{r,t}: \min I \equiv 1(2)\} = \{I \in \mathfrak{A}_{r,t}: \min I \equiv 1(2), \min I \geqq 3\} \cup \{I \in \mathfrak{A}_{r,t}: \min I = 1\}$.

The mapping $i \to i-2$ for $i = 3,\cdots,t$ maps the set $\{I \in \mathfrak{A}_{r,t}: 3 \leqq \min I \equiv 1(2)\}$ one-to-one onto $\{I^* \in \mathfrak{A}_{r,t-2}: \min I \equiv 1(2)\}$ and the mapping $I \to \{i-1 : i \in I, i \neq 1\}$ maps $\{I \in \mathfrak{A}_{r,t}: \min I = 1\}$ one-to-one onto $\{I \in \mathfrak{A}_{r-1,t-1}: \min I \equiv 1(2)\}$. □

LEMMA 5.6. *For* $1 \leqq r \leqq t$ *we have*

$$b_{r,t} = \binom{t - [\frac{1}{2}(t-r+1)]}{r}. \tag{5.11}$$

PROOF. When $r = t$ or $r = t-1$, then clearly $\#\{I \in \mathfrak{A}_{r,t}: \min I \equiv 1(2)\} = 1$ and the right side of (5.11) gives us the same value. When $r = 1$, then

$$\#\{I \in \mathfrak{A}_{1,t}: \min I \equiv 1(2)\} = \#\left\{1,3,\cdots,2\cdot\left[\frac{t+1}{2}\right]-1\right\} = \left(\frac{t+1}{2}\right)$$ and again the right side of (5.11). When $2 \leqq r \leqq t-2$, we apply (5.10). Then we obtain by induction

$$b_{r,t} = b_{r,t-2} + b_{r-1,t-1} = \binom{t-2-[\frac{1}{2}(t-r-1)]}{r} + \binom{t-1-[\frac{1}{2}(t-r+1)]}{r-1}$$

$$= \binom{t - [\frac{1}{2}(t-r+1)]}{r}. \quad \square$$

COROLLARY 5.1.

$$\#\,\mathfrak{A}_{r,t} = \binom{t - [\frac{1}{2}(t-r+1)]}{r} + \binom{t - [\frac{1}{2}(t-r+2)]}{r}. \tag{5.12}$$

PROOF. Lemma 5.4 and Lemma 5.6. □

COROLLARY 5.2.

$$A_r\left(\frac{1}{t},\cdots,\frac{1}{t}\right) = \frac{1}{t^t}\left[\binom{t - [\frac{1}{2}(t-r+1)]}{r} + \binom{t - [\frac{1}{2}(t-r+2)]}{r}\right]. \tag{5.13}$$

PROOF. $$A_r\left(\frac{1}{t},\cdots,\frac{1}{t}\right) = \sum_{I\in\mathfrak{A}_{r,t}}\prod_{i\in I}\frac{1}{t} = \frac{1}{t_r}\#\mathfrak{A}_{r,t}. \qquad \square$$

When $d^* = (1,0,1,\cdots)\in\mathbb{D}_t$ is the binary number with t-digits, with the first digit 1 which is itself an alternating sequence, then (5.12) gives us the number of alternating subsequences of r-digits of d^*. Now define

(5.14) $$\alpha_{r,m} = \max\{a_r(d)\colon d\in\mathbb{D}_m\}.$$

We cannot prove in general the following conjecture

(5.15) $$\alpha_{r,t} = a_r(d^*) \quad \text{for } t = 1,2,\cdots, r = 1,2,\cdots,t.$$

When

(5.16) $$S = \{(x_1,\cdots,x_t)\in\mathbb{R}^t\colon x_i \geqq 0, \quad \sum x_i \geqq 1\}$$

then the following conjecture implies (5.15)

(5.17) $$\max\{A_r(x_1,\cdots,x_t)\colon (x_1,\cdots,x_t)\in S\} = A_r(1/t,\cdots,1/t).$$

When $r = t$, then (5.17) reduces to the arithmetic-geometric mean inequality. The number in (5.12) is the supposed upper bound in the so called "Upper Bound Conjecture" of convex polytopes [1]. (After this paper was sent for publication, this longstanding conjecture was proved [3]). In fact the connections between finite nested coloured bipartite complete graphs and the combinatorial structure of a certain class of convex polytopes were the starting point of this investigation.

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